

# The low lying modes of triplet-condensed neutron matter and their effective theory

Paulo F. Bedaque\* and Amy N. Nicholson†

*Maryland Center for Fundamental Physics, Department of Physics,  
University of Maryland, College Park MD 20742-4111, USA*

The condensation of neutrons into a  $^3P_2$  superfluid phase occurs at densities relevant for the interior of neutron stars. The triplet pairing breaks rotational symmetry spontaneously and leads to the existence of gapless modes (angulons) that are relevant for many transport coefficients and to the star's cooling properties. We derive the leading terms of the low energy effective field theory, including the leading coupling to electroweak currents, valid for a variety of possible  $^3P_2$  phases.

## I. INTRODUCTION

While understanding the phases of QCD at various densities and temperatures is at the forefront of nuclear physics research, this goal remains an elusive one due to the nonperturbative nature of QCD at all but the highest densities and temperatures. Of particular interest for the study of the cores of neutron stars is that of nuclear matter above nuclear matter density,  $\rho_{nm}$ , at temperatures which are low relative to the Fermi energy. For low densities interacting nucleons may be used to describe the low-temperature properties of a system, while at asymptotically high densities QCD becomes deconfined and a number of possible ground states have been proposed. At moderate densities ( $\sim \rho_{nm}$ ) it is expected that neutrons will condense to form a superfluid state. The spontaneous breaking of any continuous symmetries by the condensate will lead to massless Goldstone bosons, which then dominate the low-energy properties of the system.

At approximately 1.5 times  $\rho_{nm}$ , s-wave interactions, which dominate at lower densities, become repulsive and  $^3P_2$  interactions dominate. This suggests that the order parameter for the superfluid phase in this regime will be of the form

$$\langle n^T \sigma_2 \sigma_i \overleftrightarrow{\nabla}_j n \rangle = \Delta_{ij} e^{i\alpha}, \quad (1)$$

where  $n$  are neutron field operators, the Pauli matrices act in spin space, and  $\alpha$  refers to the  $U(1)$  phase associated with spontaneously broken baryon number. Because the neutrons couple to form a spin-2 object,  $\Delta_{ij}$  is a symmetric traceless tensor. The condensate spontaneously breaks rotational invariance, thus there will be new massless modes associated with this breaking in addition to the usual superfluid phonon<sup>1</sup>. These massless modes, referred to as angulons [1], have been shown to provide a mechanism for neutrino emission in neutron stars. Recently, interest in the  $^3P_2$  phase of neutron matter and its transport properties has been rekindled by the observation of rapid cooling of the neutron star in Cassiopeia A [2], the youngest known neutron star in the Milky Way, interpreted by two groups as evidence for triplet pairing [3][4][5][6][7]. A systematic understanding of the properties of  $^3P_2$  condensed matter is necessary for sharpening this conclusion.

There are also further theoretical motivations for the calculations presented in this paper. One is that the very existence of angulons has recently been put into question [8]. Also, due to the spontaneous breaking of rotational symmetry, a large number of terms in the action are allowed by the symmetries and it does not seem possible to fix their coefficients by the usual matching procedure unless the ground state has a condensate of a special form like phase B, below. In fact, reference [1] assumes the ground state to be in phase B simply to avoid the problems that the other phases raise.

While the form of the order parameter is dictated by Eq. 1, different symmetric traceless tensors break different symmetries and there are several possible  $^3P_2$  phases. We may choose some orthonormal frame to write down three

---

\*Electronic address: bedaque@umd.edu

†Electronic address: amynn@umd.edu

<sup>1</sup> In some circumstances, like  $^3\text{He}$ , spin and orbital rotations are separate approximate symmetries and their breaking would lead to additional approximately gapless modes. In this paper we will not assume that spin and orbital rotations are separate symmetries and only the exactly gapless modes generated by the breaking of rotation symmetry (corresponding to the diagonal group of combined spin and orbital rotations) are considered.

simple symmetry breaking patterns:

$$\Delta^0 = \bar{\Delta} \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Phase A} \quad (2a)$$

$$\Delta^0 = \bar{\Delta} \begin{pmatrix} e^{2i\pi/3} & 0 & 0 \\ 0 & e^{-2i\pi/3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Phase B} \quad (2b)$$

$$\Delta^0 = \bar{\Delta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Phase C} \quad (2c)$$

In Phase A, rotational invariance is maintained in one plane, leading to two angulons associated with the breaking of rotational invariance in the remaining two planes. Phase B fully breaks rotational invariance, leading to three angulons, and was considered in [1] due to the simplicity of the effective theory for a unitary order parameter. Phase C leads to only one angulon due to the lack of a condensate in the third direction, but also contains gapless neutron modes. It is currently unclear which phase corresponds to the ground state for the relevant regions of neutron stars, and more complicated phases than the three simple ones presented here are certainly possible. Near the critical temperature, Ginsburg-Landau arguments can be applied and the form of the condensate is known to be a *real* symmetric matrix [9]. Estimating the coefficients of the Ginsburg-Landau free energy by the BCS approximation (weak coupling) one finds that phase A is favored (phase C is a close second). Strong coupling corrections to BCS reinforce this conclusion [10]. At lower temperatures the problem is more complicated, even in the BCS approximation. However, it was pointed out in [11][12] that, when mixing between  $^3P_2$  and  $^3F_2$  channels can be neglected, the relative ordering between the different  $^3P_2$  phases is independent of temperature, density and even neutron-neutron interactions. The  $^3F_2 - ^3P_2$  mixing alters this result somewhat by lifting some degeneracies[13]. In view of this uncertainty on the precise form of the condensate we will try to be as general as possible and derive the effective theory for a general *nodeless* phase (one where  $\Delta^0$  has no zero eigenvalues). This generality can be maintained only up to some point. Final explicit expressions for the numerical values of the coefficients will be given for phase A although they can be readily obtained for any other phase using the same methods.

Because it is not clear whether an effective theory may be derived for a general phase using the standard matching procedure, we choose instead the less elegant way of deriving the effective theory directly from a microscopic model by performing a derivative expansion to eliminate high momentum modes. In this way, both the form of the effective Lagrangian and the couplings may be determined simultaneously. The result may seem to depend strongly on the choice of the microscopic model but, in fact, most of the dependence is embedded in the value of the neutron gap ( $\bar{\Delta}$  above). By writing the effective theory coefficients in terms of the value of the neutron gap most of the dependence on the microscopic model disappears.

## II. MICROSCOPIC MODEL

To derive an effective theory describing the low-energy modes of neutron matter in a  $^3P_2$  condensed phase directly from QCD is not currently possible due to its nonperturbative nature. However, as we are only interested in low-energy properties near the Fermi surface it is sufficient to begin with a model which encapsulates the relevant properties. We choose a simple model which reproduces the leading order low-energy observables, the Fermi speed and the gap, consisting of two species (corresponding to spin states) of non-relativistic neutrons with an attractive, short range potential and a common chemical potential,

$$\mathcal{L} = \psi^\dagger (i\partial_0 - \epsilon(-i\nabla)) \psi - \frac{g^2}{4} \left( \psi^\dagger \sigma_i \sigma_2 \overleftrightarrow{\nabla}_j \psi^* \right) \chi_{ij}^{kl} \left( \psi^T \sigma_2 \sigma_i \overleftrightarrow{\nabla}_j \psi \right), \quad (3)$$

where  $\chi_{ij}^{kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})$  is the projector onto the  $^3P_2$  channel satisfying  $\chi_{ij}^{kl}\chi_{mn}^{ij} = \chi_{mn}^{kl}$ . There are two ways of interpreting the calculation we are about to describe. One is to take  $\epsilon(p) = p^2/2M - \mu$  (or its relativistic counterpart) and adjust the coupling  $g^2$  so the vacuum neutron-neutron  $^3P_2$  phase-shift is reproduced. That would make eq. (3) a reasonable schematic model to consider. But one can also take  $\epsilon(p) = v_F(p - k_F)$ , with  $v_F$  the Fermi velocity and  $k_F$  the Fermi momentum, and think of eq. (3) in terms of Fermi liquid theory [14, 15]. In that case, eq. (3) describes the effective theory valid for momenta  $p$  close to the Fermi surface, namely,  $|p - k_F| \ll k_F$ , while the interaction term is a combination of Landau  $f_L, g_L$  parameters describing the interaction between the fermionic quasi-particles. Thus our calculation can be seen as one link on a chain of effective field theories connecting the

phenomenology of neutron stars to “first principles” (either the nuclear forces or, more ambitiously, QCD). In any case, the important point to realize is that while the specific values of the Fermi velocity and  $g^2$  strongly influence the effective theory parameters they do so only through the value of the neutron gap. Written as a function of the neutron gap, the coefficients we will find are independent of the details of the interaction.

Since the angulons, as Goldstone bosons, correspond to spacetime-dependent rotations of the order parameter, we will start by rewriting the theory defined by eq. (3) in terms of the condensate. For that we introduce an auxiliary field,  $\Delta_{ij}$ , in the neutron pair (BCS) channel

$$\begin{aligned} S[\Delta, \psi] &= \int d^4x \left[ \psi^\dagger (i\partial_0 - \epsilon(-i\nabla)) \psi + \frac{1}{4g^2} \Delta_{ij}^\dagger \Delta_{ji} + \frac{\Delta_{ij}^\dagger}{4} \left( \psi^T \sigma_2 \sigma_i \overleftrightarrow{\nabla}_j \psi \right) - \frac{\Delta_{ji}}{4} \left( \psi^\dagger \sigma_i \sigma_2 \overleftrightarrow{\nabla}_j \psi^* \right) \right] \\ &= \int d^4x \left[ \frac{1}{4g^2} \Delta_{ij}^\dagger \Delta_{ji} + \frac{1}{2} \begin{pmatrix} \psi^\dagger & \psi \end{pmatrix} \begin{pmatrix} i\partial_0 - \epsilon(-i\nabla) & -\Delta_{ji} \sigma_i \sigma_2 \nabla_j \\ \Delta_{ij}^\dagger \sigma_2 \sigma_i \nabla_j & i\partial_0 + \epsilon(-i\nabla) \end{pmatrix} \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \right], \end{aligned}$$

where we have dropped the projectors with the understanding that the functional integration is restricted to only the  $\Delta_{ij}$  which are traceless and symmetric. We may now perform the gaussian integration over the fermions resulting in the following action,

$$S[\Delta] = \int d^4x \left[ \frac{1}{4g^2} \Delta_{ij}^\dagger \Delta_{ji} - i \text{Tr} \log \begin{pmatrix} i\partial_0 - \epsilon(-i\nabla) & -\Delta_{ji} \sigma_i \sigma_2 \nabla_j \\ \Delta_{ij}^\dagger \sigma_2 \sigma_i \nabla_j & i\partial_0 + \epsilon(-i\nabla) \end{pmatrix} \right]. \quad (4)$$

Up to now our calculation is exact. However, for a generic space-time dependent  $\Delta_{ij}$  this action is complicated and highly non-local. As we are only interested in deriving a low-energy effective theory, we can obtain a useful expression if we perform a derivative expansion of  $S[\Delta]$ . Keeping only the leading order terms in such an expansion gives a local action in which high momentum modes have been removed. We may then parametrize the auxiliary field in terms of our effective degrees of freedom, the angulons, to find the angulon dispersion relations and interactions for a given phase.

### III. EFFECTIVE THEORY

Following [16–19] we perform a derivative expansion on the logarithm in Eq. 4 by first separating the auxiliary field into its constant ground state plus spatial variations,  $\Delta(x) \rightarrow \Delta^0 + \Delta(x)$ . As outlined in detail in App. VI, this leads to the following expansion for the action,

$$\begin{aligned} S[\Delta] &= \int d^4x \left[ \frac{1}{4g^2} \Delta_{ij}^\dagger \Delta_{ji} - i \text{Tr} \log D_0^{-1}(p) \right. \\ &\quad \left. - i \int \frac{d^4p}{(2\pi)^4} \int_0^1 dz \text{tr} \sum_{n=0}^{\infty} (-z)^n \left[ D_0(p) \sum_{m=1}^{\infty} \frac{\partial_\mu^m}{m!} p_j [\delta D^{-1}(x)]_j (i\partial_{p_\mu})^m \right]^n D_0(p) p_k [\delta D^{-1}(x)]_k \right], \end{aligned} \quad (5)$$

where

$$D_0^{-1}(p) = \begin{pmatrix} p_0 - \epsilon(p) & i\Delta_{ji}^0 \sigma_i \sigma_2 p_j \\ -i\Delta_{ij}^{0\dagger} \sigma_2 \sigma_i p_j & p_0 + \epsilon(p) \end{pmatrix}, \quad [\delta D^{-1}(x)]_j = \begin{pmatrix} 0 & i\Delta_{ji}(x) \sigma_i \sigma_2 \\ -i\Delta^\dagger(x)_{ij} \sigma_2 \sigma_i & 0 \end{pmatrix}, \quad (6)$$

and  $\text{tr}$  corresponds to a trace over Gorkov indices. The first two terms are the one-loop effective potential evaluated at  $\Delta = \Delta^0$ . The remaining terms give the space-time variation of the field  $\Delta$ , which describes not only the Goldstone bosons but also other, gapped degrees of freedom. Later, we will identify  $\Delta = R(\alpha) \Delta^0 R^T(\alpha)$ , where  $R$  is an  $SO(3)$  rotation matrix; the Goldstone boson fields  $\alpha$  are the ones parametrizing the space-time dependent rotation  $R(\alpha)$ .

The leading order term in the derivative expansion contains two derivatives. This is given by the  $m = 2, n = 1$  term in eq. (5) and leads to the following action,

$$S_2[\Delta] = -\frac{i}{4} \int d^4x \int \frac{d^4p}{(2\pi)^4} \text{tr} [D_0(p) \partial_\mu \partial_\nu \delta D^{-1}(x) \partial_{p_\mu} \partial_{p_\nu} D_0(p) \delta D^{-1}(x)] , \quad (7)$$

where we have dropped the constant terms associated with the vacuum energy. These terms can be minimized for constant  $\Delta$  to determine which phase corresponds to the ground state. However, as discussed in the Introduction, there already exists extensive literature on this issue, including the effects of the non-zero coupling to interactions in the  $^3F_2$  channel [11–13].

From Eq. 7 we disentangle the dependence on the field  $\Delta$  by performing the matrix multiplication. Upon integrating by parts we find,

$$S_2[\Delta] = \int d^4x \left[ \mathcal{A}_{\mu,i,j,\nu,k,l} \partial_\mu \Delta_{ij}^\dagger \partial_\nu \Delta_{kl}^\dagger + \mathcal{B}_{\mu,i,\nu,j} [\partial_\mu \Delta \cdot \partial_\nu \Delta^\dagger]_{ij} + \mathcal{A}_{\mu,i,j,\nu,k,l}^\dagger \partial_\mu \Delta_{ji} \partial_\nu \Delta_{lk} \right], \quad (8)$$

where we have used the shorthand  $D_{ab} \equiv [D_0(p)]_{ab}$  ( $a, b$  are Gorkov indices), and the coefficients are given by

$$\begin{aligned} \mathcal{A}_{\mu,i,j,\nu,k,l} &\equiv -\frac{i}{4} \int \frac{d^4p}{(2\pi)^4} \text{tr} [\partial_{p_\mu} (D_{12} p_j) \sigma_2 \sigma_i \partial_{p_\nu} (D_{12} p_l) \sigma_2 \sigma_k], \\ \mathcal{B}_{\mu,i,\nu,j} &\equiv -\frac{i}{4} \int \frac{d^4p}{(2\pi)^4} \text{tr} [-2 \partial_{p_\mu} (D_{11} p_i) \partial_{p_\nu} (D_{22} p_j)] . \end{aligned} \quad (9)$$

The derivatives of the propagator are

$$\begin{aligned} \partial_{p_\mu} D_{11}(p) &= \frac{1}{(p_0^2 - E_p^2)^2} [(-p_0^2 - 2p_0 \epsilon(p) - E_p^2) \delta_{\mu,0} \\ &\quad + \left( v_F \frac{p_k}{p} ((p_0 + \epsilon(p))^2 - p \cdot \Delta^{0\dagger} \Delta^0 \cdot p) + 2(p_0 + \epsilon(p)) p \cdot \Delta^{0\dagger} \Delta_k^0 \right) \delta_{\mu,k}] \\ \partial_{p_\mu} D_{12}(p) &= \frac{1}{(p_0^2 - E_p^2)^2} [(2ip_0 \Delta_{ji}^0 \sigma_i \sigma_2 p_j) \delta_{\mu,0} \\ &\quad - i \Delta_{ji}^0 \sigma_i \sigma_2 \left[ 2\epsilon(p) v_F \frac{p_k p_j}{p} + (p_0^2 - E_p^2) \delta_{jk} + 2(p \cdot \Delta^{0\dagger} \Delta^0)_k \right] \delta_{\mu,k}] \\ \partial_{p_\mu} D_{22}(p) &= \frac{1}{(p_0^2 - E_p^2)^2} [(-p_0^2 + 2p_0 \epsilon(p) - E_p^2) \delta_{\mu,0} \\ &\quad + \left( v_F \frac{p_k}{p} (-(p_0 - \epsilon(p))^2 + p \cdot \Delta^{0\dagger} \Delta^0 \cdot p) + 2(p_0 - \epsilon(p)) p \cdot \Delta^{0\dagger} \Delta_k^0 \right) \delta_{\mu,k}] , \end{aligned} \quad (10)$$

with the definition  $E_p \equiv \sqrt{\epsilon(p)^2 + p \cdot \Delta^{0\dagger} \Delta^0 \cdot p}$ . We find that the coefficients for the temporal derivative terms are given by the integrals

$$\begin{aligned} \mathcal{A}_{0,i,j,0,k,l} &= \Delta_{ca}^0 \Delta_{db}^0 (\delta_{ai} \delta_{bk} - \delta_{ab} \delta_{ik} + \delta_{ak} \delta_{ib}) a_{ajbl} \\ a_{ajbl} &\equiv 2i \int \frac{d^4p}{(2\pi)^4} \frac{p_0^2 p_a p_j p_b p_l}{(p_0^2 - E_p^2)^4} = -\frac{1}{16} \int \frac{d^3p}{(2\pi)^3} \frac{p_a p_j p_b p_l}{E_p^5} \approx -\frac{M k_F}{24\pi^2 \bar{\Delta}^2} \mathcal{I}_{ajbl}^{(2)} \\ \mathcal{B}_{0,i,0,j} &= i \int \frac{d^4p}{(2\pi)^4} \frac{p_i p_j}{(p_0^2 - E_p^2)^4} ((p_0^2 + E_p^2)^2 - 4p_0^2 \epsilon(p)^2) \\ &= \frac{1}{8} \int \frac{d^3p}{(2\pi)^3} \frac{p_i p_j (2\epsilon(p)^2 + p \cdot \Delta^0 \Delta^{0\dagger} \cdot p)}{(\epsilon(p)^2 + p \cdot \Delta^0 \Delta^{0\dagger} \cdot p)^5} \approx \frac{M k_F}{6\pi^2 \bar{\Delta}^2} \mathcal{I}_{ij}^{(1)}, \end{aligned} \quad (11)$$

where we have used the fact that, for small  $\bar{\Delta}/v_F$ , the integral is dominated by the singularity at  $p = k_F$  to make the approximations,  $p \approx k_F$ ,  $\epsilon(p) \approx v_f(p - k_F)$ . Besides the derivative expansion this is the only other approximation made up to now. While the value of the neutron gap is a famously difficult quantity to compute, there is no question that the value of the neutron gap is below  $\approx 2$  MeV and is much smaller than the Fermi energy [11, 20–22]. We have also defined the remaining angular integrals as

$$\mathcal{I}_{ij\dots}^{(\alpha)}(\hat{\Delta}^\dagger \hat{\Delta}) \equiv \int \frac{d\hat{p}}{4\pi} \frac{\hat{p}_i \hat{p}_j \dots}{(\hat{p} \cdot \hat{\Delta}^\dagger \hat{\Delta} \cdot \hat{p})^\alpha}, \quad (12)$$

where  $\hat{\Delta} \equiv \Delta^0/\bar{\Delta}$  and  $\hat{p}_i = p_i/p$ . These integrals are functions of  $\hat{\Delta}^\dagger \hat{\Delta}$ , and depend on which phase is considered (to be more precise, they depend on the squares of the eigenvalues of  $\hat{\Delta}$ ) so we will postpone their evaluation until the next section.

The spatial derivative terms in the Lagrangian are given by:

$$\begin{aligned} \mathcal{A}_{a,i,j,b,k,l} &= \Delta_{mc}^0 \Delta_{nd}^0 (\delta_{ci} \delta_{dk} - \delta_{cd} \delta_{ik} + \delta_{ck} \delta_{id}) a_{abj mnl} \\ a_{abj mnl} &\equiv \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p_0^2 - E_p^2)^4} \left[ \left( p_j (2\epsilon(p) v_F \frac{p_a p_m}{p} + (p_0^2 - E_p^2) \delta_{ma} + 2p_m [\Delta^{0\dagger} \Delta^0 \cdot p]_a) - \delta_{aj} (p_0^2 - E_p^2) p_m \right) \right] \end{aligned}$$

$$\begin{aligned}
& \times \left( p_l (2\epsilon(p) v_F \frac{p_b p_n}{p} + (p_0^2 - E_p^2) \delta_{bn} + 2p_n [\Delta^{0\dagger} \Delta^0 \cdot p]_b) - \delta_{lb} (p_0^2 - E_p^2) p_n \right) \\
& \approx \frac{M k_F v_F^2}{24\pi^2 \bar{\Delta}^2} \mathcal{I}_{abjmn}^{(2)} [1 + \mathcal{O}(\bar{\Delta}^2/v_F^2)]
\end{aligned} \tag{13}$$

$$\begin{aligned}
\mathcal{B}_{a,i,b,j} &= i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p_0^2 - E_p^2)^4} \left[ v_F \frac{p_a p_i}{p} [(p_0 + \epsilon(p))^2 - 4p \cdot \Delta^\dagger \Delta \cdot p] + 2p_j (p_0 + \epsilon(p)) [p \cdot \Delta^\dagger \Delta]_a \right. \\
&+ \left. \delta_{ai} (p_0 + \epsilon(p)) (p_0^2 - E_p^2) \right] \left[ v_F \frac{p_b p_j}{p} [-(p_0 - \epsilon(p))^2 - 4p \cdot \Delta^\dagger \Delta \cdot p] + 2p_j (p_0 - \epsilon(p)) [p \cdot \Delta^\dagger \Delta]_b + \delta_{bj} (p_0 - \epsilon(p)) (p_0^2 - E_p^2) \right] \\
&\approx -\frac{M k_f v_F^2}{6\pi^2 \bar{\Delta}^2} \mathcal{I}_{abij}^{(1)} [1 + \mathcal{O}(\bar{\Delta}^2/v_F^2)] + \frac{M k_F}{\pi^2} \ln \left( \frac{\Lambda}{k_f \bar{\Delta}} \right) \delta_{ai} \delta_{bj},
\end{aligned} \tag{14}$$

where  $\Lambda$  is an ultraviolet cutoff of the order of the breakdown scale of the effective theory, namely,  $\Lambda \approx k_f \bar{\Delta}$ . This term is suppressed by  $\sim \bar{\Delta}^2/v_f^2$  compared to the remaining ones and we will subsequently drop it.

By performing the index contractions, we can find the effective action up to two derivative terms in an expansion around the point  $x = 0$ . However, the action is space-time translation invariant and its form at  $x = 0$  determines it at any other space-time point. As explained in [18, 19, 23], the effective action is then given by dropping all undifferentiated  $\delta D(x)$  and substituting  $\Delta$  for  $\Delta^0$ . The final result is that the general form for the effective theory to second order in a derivative expansion (and up to terms of order  $\mathcal{O}(\bar{\Delta}^2/v_F^2)$  and higher) is:

$$\begin{aligned}
S_2[\Delta] &= \frac{M k_F}{12\pi^2 \bar{\Delta}^2} \int d^4 x \left[ \mathcal{I}_{ij}^{(1)}(\hat{\Delta}^\dagger \hat{\Delta}) [\partial_0 \Delta \cdot \partial_0 \Delta^\dagger]_{ij} - v_F^2 \mathcal{I}_{ijkl}^{(1)}(\hat{\Delta}^\dagger \hat{\Delta}) [\partial_k \Delta \cdot \partial_l \Delta^\dagger]_{ij} \right. \\
&+ \frac{1}{2} \mathcal{I}_{ijkl}^{(2)}(\hat{\Delta}^\dagger \hat{\Delta}) \left( -2 [\hat{\Delta} \cdot \partial_0 \Delta^\dagger]_{ij} [\hat{\Delta} \cdot \partial_0 \Delta^\dagger]_{kl} + [\partial_0 \Delta^\dagger \cdot \partial_0 \Delta^\dagger]_{ij} \hat{\Delta}_{kl}^2 \right) \\
&+ \left. \frac{v_F^2}{2} \mathcal{I}_{ijklmn}^{(2)}(\hat{\Delta}^\dagger \hat{\Delta}) \left( 2 [\hat{\Delta} \cdot \partial_k \Delta^\dagger]_{ij} [\hat{\Delta} \cdot \partial_l \Delta^\dagger]_{mn} - [\partial_k \Delta^\dagger \cdot \partial_l \Delta^\dagger]_{ij} \hat{\Delta}_{mn}^2 \right) + \text{h.c.} \right], \tag{15}
\end{aligned}$$

where now, and subsequently,  $\bar{\Delta}$  is defined as

$$\hat{\Delta} = \frac{\Delta}{\bar{\Delta}}. \tag{16}$$

#### IV. RESULTS FOR PHASE A

We will now specialize to the phase in which the eigenvalues of  $\Delta^0$  are  $\{-1/2, -1/2, 1\}$ , however, the method below can be easily carried through for any other nodeless phase. We first observe that, in the case where  $\Delta^0$  has two identical eigenvalues, the integrals  $\mathcal{I}_{ij\dots}^{(\alpha)}(\hat{\Delta}^\dagger \hat{\Delta})$  can be written as

$$\begin{aligned}
\mathcal{I}_{ij}^{(\alpha)}(\hat{\Delta}^\dagger \hat{\Delta}) &= A^{(\alpha)} \delta_{ij} + B^{(\alpha)} (\hat{\Delta}^\dagger \hat{\Delta})_{ij} \\
\mathcal{I}_{ijkl}^{(\alpha)}(\hat{\Delta}^\dagger \hat{\Delta}) &= C^{(\alpha)} \delta_{ij} \delta_{kl} + D^{(\alpha)} \delta_{ij} (\hat{\Delta}^\dagger \hat{\Delta})_{kl} + E^{(\alpha)} (\hat{\Delta}^\dagger \hat{\Delta})_{ij} (\hat{\Delta}^\dagger \hat{\Delta})_{kl} + \text{perm.} \\
\mathcal{I}_{ijklmn}^{(\alpha)}(\hat{\Delta}^\dagger \hat{\Delta}) &= F^{(\alpha)} \delta_{ij} \delta_{kl} \delta_{mn} + G^{(\alpha)} \delta_{ij} \delta_{kl} (\hat{\Delta}^\dagger \hat{\Delta})_{mn} + H^{(\alpha)} \delta_{ij} (\hat{\Delta}^\dagger \hat{\Delta})_{kl} (\hat{\Delta}^\dagger \hat{\Delta})_{mn} + \text{perm.} \\
&+ J^{(\alpha)} (\hat{\Delta}^\dagger \hat{\Delta})_{ij} (\hat{\Delta}^\dagger \hat{\Delta})_{kl} (\hat{\Delta}^\dagger \hat{\Delta})_{mn} + \text{perm.},
\end{aligned} \tag{17}$$

where “+ perm.” indicates that all permutations of the indices should be included (the last term, for instance, has its 6 indices combined in all 720 possible ways). Numerical values for the coefficients  $A^{(\alpha)}, B^{(\alpha)} \dots$  are given in the appendix.

In phase A, rotation invariance is only partially broken, with invariance under rotation in the  $(x, y)$ -plane preserved. Thus, we have only two angulons,  $\alpha_{1,2}$ , in addition to the usual phonon. We may parametrize the field as

$$\Delta = e^{-i(\alpha_1(x) J_1 + \alpha_2(x) J_2)/f} \Delta^0 e^{i(\alpha_1(x) J_1 + \alpha_2(x) J_2)/f}, \tag{18}$$

where  $J_{1,2}$  correspond to the generators of infinitesimal rotations about the  $x$ - and  $y$ -axes, respectively and the “decay constant”  $f$  will be chosen later in order to simplify the expressions. Here we will only consider the effective theory for the angulons, corresponding to spontaneously broken  $SO(3)$  rotation symmetry. The theory for the phonon associated with breaking of  $U(1)$  baryon number decouples from that of the angulons and may be treated separately. The effective theory for the phonon is much simpler and its parameters can be determined by matching as done, in the context of neutron triplet pairing, in [1]. In fact, a much more general result can be obtained by general field theoretical arguments [24]. We will ignore the superfluid phonon from now on.

### A. Kinetic terms and specific heat

A derivative expansion of our Lagrangian in terms of the angulon fields to second order gives

$$\begin{aligned}
S_2[\Delta] &= \frac{1}{f^2} \frac{Mk_F}{6\pi^2\bar{\Delta}^2} \int d^4x \left[ \frac{9}{16} \left( 8A^{(1)} + 5B^{(1)} + 80C^{(2)} + 62D^{(2)} + 53E^{(2)} \right) [(\partial_0\alpha_1)^2 + (\partial_0\alpha_2)^2] \right. \\
&+ v_F^2 \left[ -\frac{9}{64} (8(32C^{(1)} + 14D^{(1)} + 5E^{(1)} + 288F^{(2)} + 162G^{(2)} + 90H^{(2)}) + 333J^{(2)}) [(\partial_y\alpha_2)^2 + (\partial_x\alpha_1)^2] \right. \\
&- \frac{9}{32} (8(16C^{(1)} + 4D^{(1)} + E^{(1)} + 96F^{(2)} + 30G^{(2)} + 9H^{(2)}) + 21J^{(2)}) [\partial_x\alpha_1\partial_y\alpha_2 + \partial_y\alpha_1\partial_x\alpha_2] \\
&- \frac{9}{8} (64C^{(1)} + 58D^{(1)} + 52E^{(1)} + 912F^{(2)} + 852G^{(2)} + 801H^{(2)} + 759J^{(2)}) [(\partial_z\alpha_1)^2 + (\partial_z\alpha_2)^2] \\
&- \left. \left. \frac{9}{64} (8(64C^{(1)} + 22D^{(1)} + 7E^{(1)} + 480F^{(2)} + 222G^{(2)} + 108H^{(2)}) + 375J^{(2)}) [(\partial_y\alpha_1)^2 + (\partial_x\alpha_2)^2] \right] \right] \\
&= \int d^4x \left[ \left( 3 + \frac{\pi}{\sqrt{3}} \right) [(\partial_0\alpha_1)^2 + (\partial_0\alpha_2)^2] + v_F^2 \left[ \left( \frac{\pi}{9\sqrt{3}} - \frac{3}{2} \right) [(\partial_z\alpha_1)^2 + (\partial_z\alpha_2)^2] \right. \right. \\
&- \frac{4\pi}{3\sqrt{3}} [(\partial_y\alpha_1)^2 + (\partial_x\alpha_2)^2] + \left( \frac{2\pi}{9\sqrt{3}} - \frac{3}{2} \right) [(\partial_y\alpha_2)^2 + (\partial_x\alpha_1)^2] \\
&+ \left. \left. \left( \frac{3}{2} - \frac{14\pi}{9\sqrt{3}} \right) [\partial_x\alpha_1\partial_y\alpha_2 + \partial_y\alpha_1\partial_x\alpha_2] \right] \right], \tag{19}
\end{aligned}$$

where in the second line we made the choice

$$f^2 = \frac{Mk_F}{6\pi^2\bar{\Delta}^2}. \tag{20}$$

Note that this action is symmetric under the interchange  $\{x, 1\} \leftrightarrow \{y, 2\}$ , in accordance with our expectation of a preserved rotation symmetry in the  $(x, y)$ -plane.

The condensate mixes the two angulons through the spatial derivative terms. The angulon dispersion relations may be found by diagonalizing the following matrix,

$$G(p) = \begin{pmatrix} ap_0^2 + v^2(bp_z^2 + cp_y^2 + dp_x^2) & ev^2p_xp_y \\ ev^2p_xp_y & ap_0^2 + v^2(bp_z^2 + cp_x^2 + dp_y^2) \end{pmatrix}, \tag{21}$$

with

$$a = 3 + \frac{\pi}{\sqrt{3}}, \quad b = -\frac{3}{2} + \frac{\pi}{9\sqrt{3}}, \quad c = -\frac{4\pi}{3\sqrt{3}}, \tag{22}$$

$$d = -\frac{3}{2} + \frac{2\pi}{9\sqrt{3}}, \quad e = \frac{3}{2} - \frac{14\pi}{9\sqrt{3}}. \tag{23}$$

$$\tag{24}$$

The values of  $p_0$  that make the determinant of  $G(p)$  vanish correspond to the poles of the angulon propagator and define their dispersion relations. As expected, the energies are proportional to the the Fermi velocity  $v_F$  times spatial momenta. But there is no expectation that the velocity of the angulons will be independent of the direction. In fact, in the particular case where the propagation is along the axis  $x, y$ , and  $z$  the corresponding velocities (for the two modes 1 and 2) are

$$\begin{aligned}
v_{x,y}^{(1)} &= \frac{v_F}{3} \sqrt{\frac{117}{18 + 2\sqrt{3}\pi}} - 2 \approx 0.477v_F, \\
v_{x,y}^{(2)} &= 2v_F \sqrt{\frac{\pi}{9\sqrt{3} + 3\pi}} \approx 0.709v_F, \\
v_z^{(1,2)} &= \frac{v_F}{3} \sqrt{\frac{99}{18 + 2\sqrt{3}\pi}} - 1 \approx 0.519v_F.
\end{aligned} \tag{25}$$

The dispersion relations for modes moving in a general direction are

$$\begin{aligned} p_0^{(1)} &= \frac{\sqrt{27\sqrt{3}|p|^2 - 2\pi[2(p_x^2 + p_y^2) + p_z^2]}}{3\sqrt{2(3\sqrt{3} + \pi)}} v_F \\ p_0^{(2)} &= \frac{\sqrt{24\pi(p_x^2 + p_y^2) + 27\sqrt{3}p_z^2 - 2\pi p_z^2}}{3\sqrt{2(3\sqrt{3} + \pi)}} v_F . \end{aligned} \quad (26)$$

The angulon modes have linear dispersion relations at small momenta, which may be used to compute the angulon contribution to the low temperature specific heat. In fact, it is given by

$$\begin{aligned} c_v &= \sum_{a=1,2} \frac{d}{dT} \int \frac{d^3p}{(2\pi)^3} \frac{\epsilon_a(p)}{e^{\epsilon_a(p)/T} - 1} \\ &\approx 16.16 \frac{T^3}{v_F^3} = 1.44 \times 10^{-13} \left( \frac{T/^\circ K}{v_F/c} \right)^3 \frac{\text{erg}}{^\circ K \text{cm}^3}, \end{aligned} \quad (27)$$

where  $\epsilon_{1,2}$  is the energy of the two uncoupled angulons. The dependence  $c_v \sim T^3/v_F^3$  follows from dimensional analysis; the numerical coefficient comes from a numerical integration. For temperatures well below the condensation temperature for neutrons in neutron stars the specific heat due to electrons dominates [25], and the angulon contribution is a few orders of magnitude smaller.

### B. Angulon interactions

The leading order effective action shown in Eq. 15 also describes interactions between angulons. Since the gapless modes, like the angulons, dominate transport processes at small temperatures, their interaction is relevant for the calculations of these quantities. The somewhat tedious process of expanding the action to quartic order in the angulon fields leads to

$$\begin{aligned} S_4[\Delta] &= \frac{1}{f^2} \int d^4x \left[ \left( 3 + \frac{\pi}{\sqrt{3}} \right) (\alpha_2^2 (\partial_0 \alpha_1)^2 + \alpha_1^2 (\partial_0 \alpha_2)^2) + \left( 12 + \frac{4\pi}{\sqrt{3}} \right) (\alpha_1^2 (\partial_0 \alpha_1)^2 + \alpha_2^2 (\partial_0 \alpha_2)^2) \right. \\ &\quad + \left( 18 + 2\sqrt{3}\pi \right) \alpha_1 \alpha_2 \partial_0 \alpha_1 \partial_0 \alpha_2 + v_F^2 \left[ \left( \frac{\pi}{9\sqrt{3}} - \frac{3}{2} \right) (\alpha_2^2 (\partial_x \alpha_1)^2 + \alpha_1^2 (\partial_y \alpha_2)^2) \right. \\ &\quad + \left( \frac{8\pi}{9\sqrt{3}} - 6 \right) (\alpha_1^2 (\partial_x \alpha_1)^2 + \alpha_2^2 (\partial_y \alpha_2)^2) - \frac{4\pi}{3\sqrt{3}} (\alpha_1^2 (\partial_x \alpha_2)^2 + \alpha_2^2 (\partial_y \alpha_1)^2) \\ &\quad + \left( 3 - \frac{28\pi}{9\sqrt{3}} \right) (\alpha_1^2 \partial_x \alpha_1 \partial_y \alpha_2 + \alpha_2^2 \partial_x \alpha_1 \partial_y \alpha_2 + \alpha_1^2 \partial_x \alpha_2 \partial_y \alpha_1 + \alpha_2^2 \partial_x \alpha_2 \partial_y \alpha_1) \\ &\quad + \left( 3 - \frac{10\pi}{3\sqrt{3}} \right) (\alpha_1 \alpha_2 \partial_x \alpha_1 \partial_y \alpha_1 + \alpha_1 \alpha_2 \partial_x \alpha_2 \partial_y \alpha_2) - \left( \frac{16\pi}{9\sqrt{3}} + 6 \right) (\alpha_1 \alpha_2 \partial_x \alpha_1 \partial_x \alpha_2 + \alpha_1 \alpha_2 \partial_y \alpha_1 \partial_y \alpha_2) \\ &\quad - \left( \frac{35\pi}{9\sqrt{3}} + \frac{3}{2} \right) (\alpha_2^2 (\partial_x \alpha_2)^2 + \alpha_1^2 \partial_y \alpha_1^2) + \left( \frac{2\pi}{9\sqrt{3}} - \frac{3}{2} \right) (\alpha_2^2 (\partial_z \alpha_1)^2 + \alpha_1^2 (\partial_z \alpha_2)^2) \\ &\quad \left. \left. - \left( \frac{\pi}{\sqrt{3}} + \frac{9}{2} \right) (\alpha_1^2 (\partial_z \alpha_1)^2 + \alpha_2^2 (\partial_z \alpha_2)^2) - \left( \frac{22\pi}{9\sqrt{3}} + 6 \right) \alpha_1 \alpha_2 \partial_z \alpha_1 \partial_z \alpha_2 \right] \right] . \end{aligned} \quad (28)$$

### C. Weak interactions

Angulons couple to electroweak currents. Since they are not electrically charged, at leading order in the Fermi constant  $G_F$  the only possible coupling is with the neutral current mediated by the Z boson. In this section we derive this coupling.

We begin by adding the following interaction terms to the microscopic Lagrangian,

$$\mathcal{L}_W = C_V Z_0^0 \psi^\dagger \psi + C_A Z_i^0 \psi^\dagger \sigma_i \psi , \quad (29)$$

where  $Z_0^0$ ,  $Z_i^0$  are the temporal and spatial components of the  $Z^0$  boson, respectively and the couplings are given by

$$C_{V,A}^2 = \tilde{C}_{V,A}^2 \frac{G_F M_Z^2}{2\sqrt{2}} , \quad (30)$$

where  $\tilde{C}_V = -1$  by vector current conservation, and  $\tilde{C}_A \sim 1.1 \pm 0.15$  [26] is given by the sum of the nucleon isovector axial coupling,  $g_A$ , and the matrix element of the strange axial-current in the proton,  $\Delta s$ . Here we choose the vacuum form of the interactions as little is known about their renormalization when modes far from the Fermi surface are removed.

The action for the angulons including the weak vertex is

$$\begin{aligned} S[\Delta] &= -i \int d^4x \text{Tr} \log[D_0^{-1} + \delta D^{-1} + C_A Z_m^0 \Sigma_m] \\ &\approx -i \int d^4x \text{Tr} (\log[D_0^{-1} + \delta D^{-1}] + (D_0^{-1} + \delta D^{-1})^{-1} C_A Z_m^0 \Sigma_m) \end{aligned} \quad (31)$$

where

$$\Sigma_m \equiv \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_2 \sigma_m \sigma_2 \end{pmatrix}, \quad (32)$$

and we have taken only the leading order in a weak coupling expansion. We may now perform a derivative expansion of the propagator using the method outlined in App. VI,

$$(D_0^{-1} + \delta D^{-1})^{-1} = \int \frac{d^4p}{(2\pi)^4} \text{tr} \left( \sum_n \left[ D_0(p) \sum_{m=1} \frac{\partial_\mu^m}{m!} p_j [\delta D^{-1}(x)]_j (i\partial_{p_\mu})^m \right]^n D_0(p) \right). \quad (33)$$

The leading order term in the derivative expansion of the weak interaction contribution to the Lagrangian is given by  $m = n = 1$ . The only non-zero terms are

$$\begin{aligned} \mathcal{L}_W[\Delta] &= C_A Z_m^0 \int \frac{d^4p}{(2\pi)^4} \text{tr} [D_0(p) \partial_0 p_j [\delta D^{-1}(x)]_j \partial_{p_0} D_0(p) \Sigma_m] \\ &= C_A Z_m^0 \int \frac{d^4p}{(2\pi)^4} p_j \text{tr} [-D_{12} \partial_0 \delta \Delta_{ij}^\dagger \sigma_2 \sigma_i \partial_{p_0} D_{11} \sigma_m + D_{11} \partial_0 \delta \Delta_{ji} \sigma_i \sigma_2 \partial_{p_0} D_{21} \sigma_m \\ &\quad - D_{22} \partial_0 \delta \Delta_{ij}^\dagger \sigma_2 \sigma_i \partial_{p_0} D_{12} \sigma_2 \sigma_m \sigma_2 + D_{21} \partial_0 \delta \Delta_{ji} \sigma_i \sigma_2 \partial_{p_0} D_{22} \sigma_2 \sigma_m \sigma_2] \\ &= C_A Z_m^0 \partial_0 \delta \Delta_{ij}^\dagger \Delta_{kl} \int \frac{d^4p}{(2\pi)^4} i p_j p_k \text{tr} \left[ \frac{-1}{p_0^2 - E_p^2} \sigma_l \sigma_2 \sigma_2 \sigma_i \frac{-p_0^2 - 2p_0 \epsilon_p - E_p^2}{(p_0^2 - E_p^2)^2} \sigma_m \right. \\ &\quad \left. + \frac{p_0 - \epsilon_p}{p_0^2 - E_p^2} \sigma_2 \sigma_i \frac{2p_0 \sigma_l \sigma_2}{(p_0^2 - E_p^2)^2} \sigma_2 \sigma_m \sigma_2 \right] + \text{h.c.} \\ &= 2\epsilon_{lim} C_A Z_m^0 \partial_0 \delta \Delta_{ij}^\dagger \Delta_{kl} \int \frac{d^4p}{(2\pi)^4} p_j p_k \frac{-p_0^2 + 4p_0 \epsilon_p + E_p^2}{(p_0^2 - E_p^2)^3} + \text{h.c.} \\ &= \frac{i}{2} \epsilon_{lim} C_A Z_m^0 \partial_0 \delta \Delta_{ij}^\dagger \Delta_{kl} \int \frac{d^3p}{(2\pi)^3} \frac{p_j p_k}{E_p^3} + \text{h.c.} \\ &\approx \epsilon_{lim} C_A Z_m^0 i \partial_0 \delta \Delta_{ij}^\dagger \Delta_{kl} \frac{M k_F}{2\pi^2 |\Delta|^2} \mathcal{I}_{jk}^{(1)} + \text{h.c.} \\ &\rightarrow 3f^2 \epsilon_{lim} C_A Z_m^0 i \partial_0 \Delta_{ij}^\dagger \Delta_{kl} \mathcal{I}_{jk}^{(1)} + \text{h.c.}, \end{aligned} \quad (34)$$

where in the last step we used the translation invariance of the effective lagrangian, as explained previously when deriving the strong interactions. Expanding this in terms of the angulon fields  $\alpha$  gives the leading contribution to the action from the angulon-neutral current vertex

$$\begin{aligned} S_W[\Delta] &= C_A \int d^4x \left[ \frac{27}{8} (4A^{(1)} + 3B^{(1)}) f(Z_2^0 i \partial_0 \alpha_2 - Z_1^0 i \partial_0 \alpha_1) + \frac{27}{8} (4A^{(1)} + 3B^{(1)}) Z_3^0 (\alpha_2 i \partial_0 \alpha_1 - \alpha_1 i \partial_0 \alpha_2) + \dots \right] \\ &= C_A \int d^4x [9f(Z_2^0 i \partial_0 \alpha_2 - Z_1^0 i \partial_0 \alpha_1) + 9Z_3^0 (\alpha_2 i \partial_0 \alpha_1 - \alpha_1 i \partial_0 \alpha_2) + \dots]. \end{aligned} \quad (35)$$

## V. SUMMARY

We have derived a low-energy effective theory describing the Goldstone bosons associated with broken rotational symmetry in a  $^3P_2$  condensed neutron superfluid (angulons). Because transport properties are dominated by the



low lying excitation modes, this theory provides a link connecting the theory of nuclear forces to many quantities of interest in neutron star phenomenology. Since there is still controversy as to which of the many  ${}^3P_2$  phases are realized in Nature we have tried to keep our calculation as general as possible. Ultimately, however, the numerical value of the coefficients of the effective action do depend on the particular  ${}^3P_2$  phase and we give explicit values for the “phase A” as defined in Eq. 2. This effective theory is valid for angulon energies below the energy scale  $\sim 2k_f\bar{\Delta}$  where other degrees of freedom, like unpaired neutrons, appear. For thermodynamical calculations, the theory is valid for temperatures below the scale  $\sim 2k_f\bar{\Delta}$ . A simple application of the effective theory, the calculation of the angulon contribution to the specific heat, was discussed. We also considered the coupling of angulons to neutral currents, since quantities like neutrino opacity and emission rates depend on this coupling, and gave an explicit form for the angulon-angulon-Z vertex.

A series of improvements and extensions to the effective theory discussed here are desirable. For applications to neutron stars, we should consider the presence of both protons and neutrons. The protons are superconducting and lead only to another gapped mode but they are important in mediating the interaction between angulons (with whom they interact through strong forces) and the gapless electron (with whom they interact electromagnetically). In our microscopic action for neutrons we have included only the dominant forces leading to  ${}^3P_2$  pairing. While expected to be repulsive and weaker, neutron interactions in other channels can have an influence on the angulon effective theory. It would be very desirable to quantify this effect. We have not given much attention to the gapped modes corresponding to a change in the eigenvalues of  $\Delta$ . While their importance is exponentially suppressed at small temperatures they can be numerically important at temperatures of relevance to some stages of neutron star evolution. Our method of deriving the effective theory by performing a derivative expansion on a microscopic theory allows us to address this question and we plan to come back to it in a future publication. Finally, the energy difference between different  ${}^3P_2$  phases is small. In particular, the condensation energy of phase C in Eq. 2 is only a few percent above that for phase A. This restricts the validity of the effective theory somewhat and it would be important to quantify the importance of the other nearby minima to the low energy physics of the system.

## VI. APPENDIX: THE DERIVATIVE EXPANSION

In order to set up the derivative expansion of

$$\text{Tr} \log D^{-1}(i\partial, x) = \text{Tr} \log \underbrace{[D_0^{-1}(i\partial) + \delta D^{-1}(i\partial, x)]}_{D^{-1}(i\partial, 0)} \quad (36)$$

we first use the relation

$$\begin{aligned} \text{Tr} \ln(A + B) &= \text{Tr} \ln A + \text{Tr} \ln(1 + A^{-1}B) = \text{Tr} \ln A + \text{Tr} \ln(1 + BA^{-1}) \\ &= \text{Tr} \ln A + \sum_{n=0}^{\infty} \text{Tr} \frac{1}{n+1} (A^{-1}B)^{n+1} \\ &= \text{Tr} \ln A + \text{Tr} \int_0^1 dz (1 + zA^{-1}B)^{-1} A^{-1}B \\ &= \text{Tr} \ln A + \text{Tr} \int_0^1 dz (A + zB)^{-1} B \end{aligned} \quad (37)$$

to find

$$\text{Tr} \log D_0^{-1} + \int_0^1 dz \text{Tr} \frac{1}{D_0^{-1} + z\delta D^{-1}}. \quad (38)$$

The second term contains the dependence on the space-time variation of  $\Delta$ . This term is complicated to compute because it contains  $\partial$  and  $x$  which do not commute. One trick to deal with this is to substitute in the integrand  $\partial \rightarrow ip, x \rightarrow x + i\partial_p$  and integrate over  $p$  [27]. To do this, we first need to find the inverse of the operator  $D_0^{-1} + z\delta D^{-1}$  in terms of  $p, \partial_p$ , so we look at

$$[D_0^{-1} + z\delta D^{-1}]^{-1} \equiv G(x, y) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot y} G(p, i\partial_p) e^{-ip \cdot x}. \quad (39)$$

Using

$$(D_0^{-1} + z\delta D^{-1}) G(x, y) = \delta^4(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot y} (D_0^{-1} + z\delta D^{-1}) G(p, i\partial_p) e^{-ip \cdot x}, \quad (40)$$

we find

$$G(p, i\partial_p) = \left[ D_0^{-1}(p) + zp_j [\delta D^{-1}(i\partial_p)]_j \right]^{-1}, \quad (41)$$

where

$$\begin{aligned} D_0^{-1}(p) &= \begin{pmatrix} p_0 - \epsilon(p) & ip_j \Delta_{ji}^0 \sigma_i \sigma_2 \\ -ip_j \Delta_{ij}^{0\dagger} (\sigma_2 \sigma_i) & p_0 + \epsilon(p) \end{pmatrix}, \\ [\delta D^{-1}(i\partial_p)]_j &= \begin{pmatrix} 0 & i\Delta_{ji}(i\partial_p) \sigma_i \sigma_2 \\ -i\Delta_{ij}^\dagger(i\partial_p) \sigma_2 \sigma_i & 0 \end{pmatrix}. \end{aligned} \quad (42)$$

We may now expand the second term in Eq. 38 as

$$\begin{aligned} & \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dz \text{tr} \left[ D_0^{-1}(p) + zp_j [\delta D^{-1}(i\partial_p + x)]_j \right]^{-1} p_k [\delta D^{-1}(x)]_k \\ &= \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dz \text{tr} \left[ D_0^{-1}(p) + zp_j [\delta D^{-1}(x)]_j + zp_j \sum_{m=1} \frac{(i\partial_{p_\mu})^m}{m!} \partial_\mu^m [\delta D^{-1}(x)]_j \right]^{-1} p_k [\delta D^{-1}(x)]_k \\ &= \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dz \text{tr} \left[ 1 + zp_j (D_0^{-1}(p) + zp_k [\delta D^{-1}(x)]_k)^{-1} \sum_{m=1} \frac{(i\partial_{p_\mu})^m}{m!} \partial_\mu^m [\delta D^{-1}(x)]_j \right]^{-1} \\ &\times \left[ D_0^{-1}(p) + zp_j [\delta D^{-1}(x)]_j \right]^{-1} p_k [\delta D^{-1}(x)]_k \\ &= \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dz \text{tr} \sum_{n=0}^{\infty} (-z)^n \left[ D_0(p) \sum_{m=1} \frac{\partial_\mu^m}{m!} p_j [\delta D^{-1}(x)]_j (i\partial_{p_\mu})^m \right]^n D_0(p) p_k [\delta D^{-1}(x)]_k, \end{aligned} \quad (43)$$

## VII. APPENDIX: COEFFICIENTS FOR THE INTEGRALS $\mathcal{I}_{ij\dots}^{(\alpha)}$

Here we show how to compute the numerical coefficients  $A^{(\alpha)}, B^{(\alpha)}, \dots$  appearing in Eq. 17. Consider, for instance,  $\mathcal{I}_{ij}^{(1)}$ . We first multiply the upper equation in Eq. 17 by  $(\hat{\Delta}^\dagger \hat{\Delta})_{ij}$  and  $\delta_{ij}$  to obtain

$$\begin{aligned} \underbrace{\int \frac{d\hat{p}}{4\pi} \frac{1}{\hat{p} \cdot (\hat{\Delta}^\dagger \hat{\Delta}) \cdot \hat{p}}}_{\frac{4\pi}{3\sqrt{3}}} &= 3A^{(1)} + \underbrace{\text{tr}(\hat{\Delta}^\dagger \hat{\Delta})}_{3/2} B^{(1)} \\ \underbrace{\int \frac{d\hat{p}}{4\pi}}_1 &= \underbrace{\text{tr}(\hat{\Delta}^\dagger \hat{\Delta})}_{3/2} A^{(1)} + \underbrace{\text{tr}(\hat{\Delta}^\dagger \hat{\Delta})^2}_{9/8} B^{(1)}. \end{aligned} \quad (44)$$

Solving this system of equations we find

$$A^{(1)} = \frac{4}{3} \left( \frac{\pi}{\sqrt{3}} - 1 \right), \quad B^{(1)} = \frac{8}{3} - \frac{16\pi}{9\sqrt{3}} \quad (45)$$

The same method can be easily implemented in computer algebra packages and we find

$$\begin{aligned} C^{(1)} &= -\frac{4}{27} + \frac{145\pi}{1458\sqrt{3}}, & D^{(1)} &= \frac{14}{27} - \frac{220\pi}{729\sqrt{3}}, & E^{(1)} &= -\frac{10}{27} + \frac{152\pi}{729\sqrt{3}} \\ C^{(2)} &= \frac{11}{36} - \frac{25\pi}{243\sqrt{3}}, & D^{(2)} &= -\frac{10}{9} + \frac{128\pi}{243\sqrt{3}}, & E^{(2)} &= \frac{8}{9} - \frac{112\pi}{243\sqrt{3}} \\ F^{(2)} &= \frac{43}{1080} - \frac{263\pi}{13122\sqrt{3}}, & G^{(2)} &= -\frac{59}{270} + \frac{256\pi}{2187\sqrt{3}}, & H^{(2)} &= \frac{16}{45} - \frac{424\pi}{2187\sqrt{3}}, & J^{(2)} &= -\frac{8}{45} + \frac{640\pi}{6561\sqrt{3}}. \end{aligned} \quad (46)$$

## Acknowledgments

This work was supported in part by U.S. DOE grant No. DE-FG02-93ER-40762.

- 
- [1] P. F. Bedaque, G. Rupak, and M. J. Savage, Phys.Rev. **C68**, 065802 (2003), nucl-th/0305032.
  - [2] C. O. Heinke and W. C. Ho, Astrophys.J. **719**, L167 (2010), 1007.4719.
  - [3] D. Page, M. Prakash, J. M. Lattimer, and A. W. Steiner, Phys.Rev.Lett. **106**, 081101 (2011), 1011.6142.
  - [4] P. S. Shternin, D. G. Yakovlev, C. O. Heinke, W. C. Ho, and D. J. Patnaude, Mon.Not.Roy.Astron.Soc. **412**, L108 (2011), 1012.0045.
  - [5] D. G. Yakovlev, W. C. Ho, P. S. Shternin, C. O. Heinke, and A. Y. Potekhin, Mon.Not.Roy.Astron.Soc. **411**, 1977 (2011), 1010.1154.
  - [6] D. Page (2012), 1206.5011.
  - [7] D. Page, M. Prakash, J. M. Lattimer, and A. W. Steiner (2011), 1110.5116.
  - [8] L. Leinson, Phys.Rev. **C85**, 065502 (2012), 1206.3648.
  - [9] R. Richardson, Phys.Rev. **D5**, 1883 (1972).
  - [10] V. Vulovic and J. Sauls, Phys.Rev. **D29**, 2705 (1984).
  - [11] T. Takatsuka and R. Tamagaki, Prog.Theor.Phys.Suppl. **112**, 27 (1993).
  - [12] V. Khodel, V. Khodel, and J. W. Clark, Phys.Rev.Lett. **81**, 3828 (1998), nucl-th/9807034.
  - [13] V. Khodel, J. W. Clark, and M. Zverev, Phys.Rev.Lett. **87**, 031103 (2001), nucl-th/0101045.
  - [14] S. Backman, G. Brown, and J. Niskanen, Phys.Rept. **124**, 1 (1985).
  - [15] A. Migdal, *Theory of finite Fermi systems, and applications to atomic nuclei*, Interscience monographs and texts in physics and astronomy (Interscience Publishers, 1967), URL <http://books.google.com/books?id=M-NEAAAAIAAJ>.
  - [16] C. Fraser, Z.Phys. **C28**, 101 (1985).
  - [17] O. Cheyette, Phys.Rev.Lett. **55**, 2394 (1985).
  - [18] O. Cheyette (1987).
  - [19] L. Chan, Phys.Rev.Lett. **54**, 1222 (1985).
  - [20] M. Baldo, O. Elgarøy, L. Engvik, M. Hjorth-Jensen, and H.-J. Schulze, Phys. Rev. C **58**, 1921 (1998), URL <http://link.aps.org/doi/10.1103/PhysRevC.58.1921>.
  - [21] L. Amundsen and E. Ostgaard, Nucl.Phys. **A442**, 163 (1985).
  - [22] W. Zuo, C. X. Cui, U. Lombardo, and H.-J. Schulze, Phys. Rev. C **78**, 015805 (2008), URL <http://link.aps.org/doi/10.1103/PhysRevC.78.015805>.
  - [23] A. K. Das and M. B. Hott, Phys.Rev. **D50**, 6655 (1994), hep-ph/9407283.
  - [24] D. Son (2002), hep-ph/0204199.
  - [25] D. Yakovlev, K. Levenfish, and Y. Shibano, Phys.Usp. **42**, 737 (1999), astro-ph/9906456.
  - [26] M. J. Savage and J. Walden, Phys.Rev. **D55**, 5376 (1997), hep-ph/9611210.
  - [27] L.-H. Chan, Phys. Rev. Lett. **54**, 1222 (1985), URL <http://link.aps.org/doi/10.1103/PhysRevLett.54.1222>.